[13] Rust, B. "A Simple Algorithm for Computing the Generalized Inverse of a Matrix," *Communications of the Association of Computing Machinery*, 9 (1966), 381-387.
[14] Shepard, R. N. "The Analysis of Proximities: Multidimensional Scaling with an Unknown Distance Function," *Psychometrika*, 27 (1962), 125-139.
[15] ———. "Analysis of Proximities as a Technique for the Study of Information Processing in Man," *Human Factors*, 5 (1963), 19-34.
[16] Tucker, L. R. "Systematic Differences Between Individuals in Perceptual Judgements," in Maynard W. Shelly and Glenn L. Bryan (eds.), *Human Judgements and Optimality*. New York: John Wiley and Sons, Inc., 1964.
[17] White R. "Space Preference and Migration: A Multi-Dimensional Analysis of the Spatial Preferences of British School Leavers, 1966." Unpublished M.S. Thesis, The Pennsylvania State University, 1967.
[18] Whitehead, A. N. *Science and Philosophy*. Paterson, New Jersey: Littlefield Adams and Co., 1964.
[19] Wilson, A. "The Impact of Climate on Industrial Growth—Tucson, Arizona: A Case Study," Chapter 17 in W. R. Derrick Sewell (ed.), *Human Dimensions of Weather Modification*. Chicago: University of Chicago Department of Geography Research Series, No. 105, 1966.

OF MAPS AND MATRICES†

Waldo R. Tobler*

There are now a number of instances in which geographical data have been collected at regular spatial intervals, for the obvious reason that numerous analytical investigations are thereby greatly facilitated. As a consequence it is not unrealistic to assume here that a measure of some geographical event has been taken at regular intervals in two orthogonal directions in a region sufficiently small to allow earth curvature to be neglected. For convenience it is assumed that the spacing $\Delta X = \Delta Y = 1$ and that the region of observations is rectangular in shape. With these assumptions the data can be arrayed in the form of a geographical matrix $G = [g_{ij}]$. Contouring of this matrix might lead to a conventional isarithmic map so that the data array can be considered to yield either a matrix or a geographical map.

The objective is now to demonstrate that matrix multiplication can be applied to the map to yield geographically useful results. Specifically, operations of the form $AGB = G^*$ will be examined. By assumption A and B are square matrices and must conform with G. The resultant of the multiplication is a new map G^*. Of particular interest are reversible processes, for which A and B have inverses, that is, $A^{-1}G^*B^{-1} = G$.

From the definition of matrix multiplication it is clear that pre-multiplication of G by A operates to affect the columns of G, and that post-multiplication by B operates on the rows of G. Since the process is linear, the order of the multiplication is immaterial. In most, but not all, cases of interest $A \equiv B$, assuming G to be square.

In general the given matrix multiplication defines a function which transforms one map into another. This function may have as many arguments as there are elements in the original matrix. A *local operator* is a function which defines a value for each element of a transform in terms of the corresponding element in the original and a small set of its neighbors. (See Rosenfeld and Pfaltz [10].) Such an operation, for example, can be defined using a neighborhood which consists of a given element and its eight immediate neighbors. In this case the function has only nine elements and is of the form

$$g^*_{ij} = f(g_{i-1,j-1}, g_{i-1,j}, g_{i-1,j+1}, g_{i,j-1}, g_{i,j}, g_{i,j+1}, g_{i+1,j-1}, g_{i+1,j}, g_{i+1,j+1}) \, .$$

Neighborhoods which are larger or of different shape can be defined in a similar manner. From the definition of matrix multiplication it follows that each g^*_{ij}

† The assistance of Dr. Henry Pollack, geophysicist in the Department of Geology at the University of Michigan, in deriving the finite difference forms of the differential equations, is greatly appreciated.

* The author is Associate Professor in the Department of Geography at the University of Michigan.

is obtained as a weighted linear combination of the corresponding g_{ij} and some (perhaps all) of its neighbors. The matrix multiplication is therefore equivalent to a class of *linear* local operators. Linear weighting functions are also known to operate as discrete (or digital) frequency filters and the matrix multiplication can also be interpreted from this point of view. (See Holloway [5] and Mesko [7].)

As a first example let G represent elevations taken from a topographic map, and let

$$A = B = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \cdots 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 \\ \vdots & & & & & & & & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 \cdots 0 & \frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

Interpreted as a local operator, the system $AGA = G^*$ is clearly equivalent to

$$g^*_{ij} = \frac{\sum_{p=-k}^{p=+k} \sum_{q=-k}^{q=+k} w_{pq} g_{i+p,\, j+q}}{\sum_{p=-k}^{p=+k} \sum_{q=-k}^{q=+k} w_{pq}}$$

with $k = 1$, and weights combined from the row and column values as follows

$$w_{pq} = \begin{array}{c|ccc} & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \hline \frac{1}{4} & \frac{1}{16} & \frac{1}{8} & \frac{1}{16} \\ \frac{1}{2} & \frac{1}{8} & \frac{1}{4} & \frac{1}{8} \\ \frac{1}{4} & \frac{1}{16} & \frac{1}{8} & \frac{1}{16} \end{array} = \begin{array}{ccc} w_{i+1,j-1} & w_{i+1,j} & w_{i+1,j+1} \\ w_{i,j-1} & w_{i,j} & w_{i,j+1} \\ w_{i-1,j-1} & w_{i-1,j} & w_{i-1,j+1} \end{array}$$

For $k > 1$ the matrix is simply less sparse. The specific a_{ij} values chosen yield a simple binomially weighted moving average so that the resulting matrix G^* now represents elevations as might be obtained through map generalization. As can be seen from the accompanying figure, the effect of the transform is

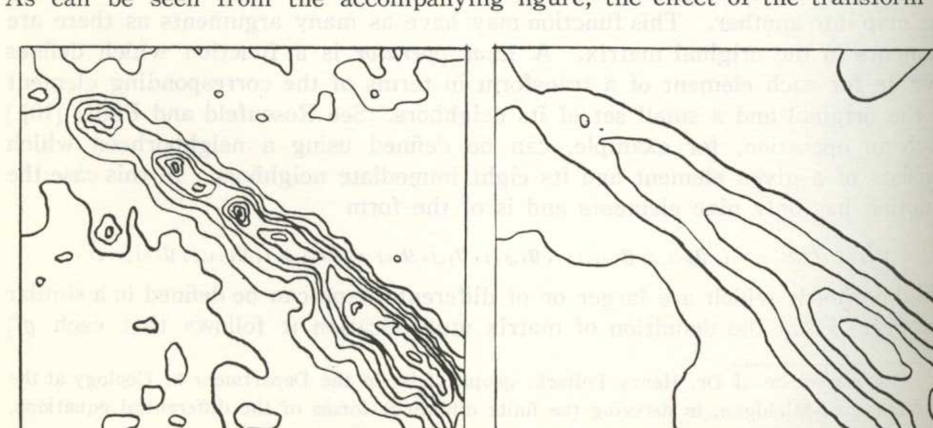

FIGURE 1: Original and Filtered Spatial Series Shown in Contour Map Form.

to simplify the contours; in other words, a low-pass spatial frequency filter has been applied. In terms of geographical theory, it can be argued that the processes under study contain small scale disturbances. These minor fluctuations distract from our ability to comprehend the general phenomena and are, therefore, appropriately eliminated. This example might perhaps be of more value if g_{ij} had represented empirical urban land values which are to be compared to some theoretical model. In this case the smoothing procedure might facilitate recognition of geographical patterns by filtering out "random" disturbances. If the elements of G had been complex numbers the operation $AGA = G^*$ could be interpreted as the smoothing of a vector field, causing small eddies to be eliminated. Interpretation of a repeated application, A^kGA^k, should be obvious.

It is known that certain linear differential equations can be written in matrix form (Lanczos [6]). The correspondence is not quite complete, however, unless boundary conditions are specified. (These appear in the matrix A, above, as slightly different weights in the corners, and correspond to special definitions required for local operators in the vicinity of the edge of the domain.) As one example consider the diffusion equation for a homogeneous medium from classical physics. It suffices here to treat the one dimensional case since it has already been demonstrated that the matrix multiplication can be decomposed into row effects and column effects.

The diffusion equation is

$$\frac{\partial g}{\partial t} = \frac{1}{\alpha} \frac{\partial^2 g}{\partial x^2}$$

where g is the quantity being diffused (e. g., heat), t is time, x is the spatial coordinate, and α is the diffusivity. In forward finite difference form, with superscripts to denote time and subscripts to denote the spatial variable, one obtains

$$g_i^{t+1} = M(g_{i-1}^t + g_{i+1}^t) + (1 - 2M)g_i^t ,$$

where $M = \Delta t/(\alpha(\Delta x)^2)$ is known as the modulus. It can now be demonstrated that the previous smoothing operator is in fact a diffusion operator if and only if, in the three weight case ($w_i > 0$),

$$\frac{w_1}{\sum_{i=-1}^{i=+1} w_i} = \frac{w_{-1}}{\sum_{i=-1}^{i=1} w_i} = M , \quad \text{and} \quad \frac{w_0}{\sum_{i=-1}^{i=1} w_i} = 1 - 2M .$$

For a numerically stable solution it is required (Todd [11], Carslaw and Jaeger [3; Ch. 18]) that $M \leq \frac{1}{2}$. In the case illustrated the equality holds and the equations are satisfied. In practice, solutions are obtained in a sequential mode rather than in the parallel, matrix computation.

The inverse operation is equivalent to backward time integration of the diffusion equation. In the finite difference form one obtains

$$g_i^{t-1} = -M(g_{i-1}^t + g_{i+1}^t) + (1 + 2M)g_i^t .$$

With the same modulus as before the inverse weights are $-\frac{1}{2}, 2, -\frac{1}{2}$. The matrix A also has an inverse (See Appendix) which however is not a local operator but has as many arguments as in the original matrix. The difference between the two inverses probably lies in the finite difference quantization process.

If A is such that $\sum_j a_{ij} = 1$ and B is such that $\sum_i b_{ij} = 1$ then it appears that A and B can be considered Markovian transition matrices, and the operation $AGB = G^*$ converts the state matrix G into a new state matrix. If $A = B^T$ the process can be considered a symmetrical Markov mesh. This line of inquiry appears to offer promise.

The simple model $AGB = G^*$ is thus seen to be related to local operators, frequency filter theory, and differential equations. Each of these topics has applications far too numerous to cite here. Only Hägerstrand's Monte Carlo simulation of the spread of innovations [4] is explored here. This model is not a classical diffusion model (the principle of conservation does not hold) but rather a model of spread and growth. (See Rapoport [9].)

Hägerstrand's simulation has a number of attractive features, including a clear exposition of the process of information spread. The intent here is to explore the extent to which the model can be reformulated as

$$G_{t+m} = A^m G_t B^m ,$$

where m denotes the generation and the superscripts on the right are also exponents. Hägerstrand himself speaks of a "neighborhood effect" and constructs a "mean information field" which has some of the attributes of weightings employed for local operators.

A set of one-dimensional weights which closely correspond to Hägerstrand's mean information field is

$$.0980 \quad .1735 \quad .6656 \quad .1735 \quad .0980 \, .$$

There are five arguments here but this is not critical. More importantly, if this set of figures is employed as a weighting, the initial number of "carriers" is reduced. This will always be the case unless the central weight satisfies $w_0 \geq 1$. One can arbitrarily choose weights which appear to approach Hägerstrand's model, e.g., $\frac{1}{3}\ \frac{4}{3}\ \frac{1}{3}$, and can verify their effect by examining the response of a unit impulse to this set of weights. It is convenient to require that the sum of the weights satisfy $\sum w = 2$ since this yields a geometrical growth in the number of knowers or carriers. If p_{ij} represents the number of potential carriers in cell i, j it is obviously easy on a computer to constrain g^*_{ij} after each iteration to satisfy $g^*_{ij} \leq p_{ij}$ but this is not conveniently incorporated into the matrix model.

The specific weights suggested above satisfy the requirements but are still capriciously chosen. The set $\frac{1}{4}, \frac{6}{4}, \frac{1}{4}$, would also have worked. Perhaps the model could be written as the finite difference analogue of the diffusion equation with sources. On the other hand,

$$g_i^{t+1} = M(g_{i-1}^t + g_{i+1}^t) + (1 - 2M)g_i^t + g_i^t ,$$

appears to come close to the model although this does not seem to correspond to any particular differential equation. The inverse is not known and it is not clear what convergence and stability mean since there is no direct differential analogue. Another approach might be to fit the weights, using the constraints $w_0 \geq 1$, $\sum w = 2$, from empirical data as was done by Hägerstrand.

In the two dimensional case the choice of individual weights is slightly more complicated but the central weight should still be greater than or equal to unity, and the entire set of weights should still sum to two for a geometrical growth rate. An asymmetric weighting can be chosen to incorporate tendencies for easier movement in particular directions. Such an effect might, for example, be of interest for the spread of a botanical or biological species under prevailing wind conditions. Exterior and interior barriers can be handled in a similar manner by appropriate choice of specific weights. For a reflecting barrier (See Nystuen [8]), set $\sum w = 2$, $w_{i+k} = 0$, when the barrier lies at the near edge of cell k; for an absorbing barrier, set $\sum w < 2$, $w_{i+k} = 0$; and for a semi-permeable barrier, reduce the appropriate w_{i+k} and choose the sum of the weights to dispose of the quantity not passed, in either a reflective or absorbing mode. Favorable (or unfavorable) environmental conditions at a location can also be incorporated by either increasing the sum of the weights (to increase the rate of growth), or by enlarging the neighborhood (making the mean information field less "steep").

The model sketched above might, on the average, yield results which approximate those of Hägerstrand. A clear disadvantage is that one must accept the notion of fractional knowers. The weights, however, could also be chosen in a stochastic manner. With $\sum w = 2$ and $w_0 = 1$ the additional unit weight can be chosen to yield integer contacts (i.e., $A = I + S$, where S consists of ones—or zeros—whose location corresponds to the probability density function of the mean information field). The matrices A and B could change with each generation. This is perhaps nothing more than an alternate computational alogorithm for Hägerstrand's model. The process is not reversible unless one knows the inverses of each of the stochastic matrices.

Another differential equation which appears promising as a starting point is

$$\frac{\partial g}{\partial t} = Mg(K - g) ,$$

where $K(=p_{ij})$ is the potential number of carriers. This has the advantage of automatically producing logistic-type growth curves. Extension to two-dimensional finite difference form is straight-forward.

Epidemiological models of the spread of diseases (Baily [1] and Brown [2]) often deal with several categories of carriers. It may be possible to construct such a model along the lines suggested here by requiring operations on one matrix to depend on a previous operation on another matrix. Another, somewhat similar, modification recognizes the eventual spatial decay of an innovation (or fad). Today's fashion in Nebraska may once have been popular in New York but has now passed out of style there. A direct finite difference form of the physical wave equation

$$\frac{\partial^2 g}{\partial t^2} = k \frac{\partial^2 g}{\partial x^2}$$

leads to

$$g_i^{t+1} = M(g_{i-1}^t + g_{i+1}^t) + 2(1 - M)g_i^t - g_i^{t-1} ,$$

which contains a lagged term. It appears that a similar effect can be achieved when $w_i > w_0$. Another modification might allow the w_i (and thus $\sum_i w_i$) to be functions of time. Even without a full development of these topics it should be clear that the model described is quite flexible.

REFERENCES

[1] Baily, N. *The Mathematical Theory of Epidemics*. New York: Hafner, 1957.

[2] Brown, L. "Models for Spatial Diffusion Research: A Review," *Technical Report Number 3*, Spatial Diffusion Study, Department of Geography, Northwestern University, ONR Task No. 389-140 Nonr 1228 (33), 1965.

[3] Carslaw, H. and J. Jaeger. *Conduction of Heat in Solids*, (2nd Ed.). Oxford: Clarendon Press, 1959.

[4] Hägerstrand, T. "A Monte Carlo Approach to Diffusion," *European Journal of Sociology*, 6 (1965), 43-67.

[5] Holloway, J. "Smoothing and Filtering of Time Series and Space Fields," *Advances in Geophysics*, 4 (1958), 351-389.

[6] Lanczos, C. *Linear Differential Operators*. New York: Van Nostrand, 1961.

[7] Mesko, C. "Two-Dimensional Filtering and the Second Derivative Method," *Geophysics*, 31 (1966), 606-617.

[8] Nystuen, J. "Boundary Shapes and Boundary Problems," *Papers, Peace Research Society*, (1967) forthcoming.

[9] Rapoport, A. "The Diffusion Problem in Mass Behavior," *General Systems*, 1 (1956) 48-55.

[10] Rosenfeld, A. and J. Pfaltz. "Sequential Operations in Digital Picture Processing," *Journal of the Association for Computing Machinery*, 13 (1966), 471-494.

[11] Todd, J. (ed.). *Survey of Numerical Methods*. New York: McGraw-Hill, 1962.

APPENDIX: THE INVERSE OF THE SMOOTHING MATRIX

Definitions

(1) A matrix $A = [a_{ij}]$ is *diagonally dominant* if

$$|a_{ii}| \geq \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|, \quad 1 \leq i \leq n .$$

(2) A matrix A is reducible, $n \geq 2$, if there exists an n by n permutation matrix P such that

$$PAP^T = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},$$

where A_{11} is an r by r submatrix, A_{22} is an $(n\text{-}r)$ square submatrix, $1 \leq r \leq n$. If such a P does not exist then A is *irreducible*.

(3) A directed graph is *strongly connected* if for any ordered pair of nodes N_j and N_k there exists a directed path $N_jN_{m1}, N_{m1}N_{m2}, \ldots, N_{mr}N_k$ connecting N_j and N_k.

(4) A matrix A is *irreducibly diagonally dominant* if A is irreducible and diagonally dominant with the strict inequality valid for at least one i.

Theorems

(I) A matrix is irreducible if, and only if, its directed graph is strongly connected.

(II) A matrix which is irreducibly diagonally dominant is non-singular. These theorems are proven in standard works on matrix algebra.

AN EXTENSION OF THE HORTON COMBINATORIAL MODEL TO REGIONAL HIGHWAY NETWORKS

Peter Haggett*

1. INTRODUCTION

Although transportation networks of all modes can be shown to share certain common mathematical properties, current research appears to fall into a number of parallel efforts, each restricted to the problems of empirically-defined media. Thus the growing body of work on traffic assignment models (U. S. Dept. of Commerce [22], Road Research Laboratory [13]) for highway networks contains rather few cross-references to hydrological research on flood-routing problems (Dawdy and O'Donnell [5] and Shen [16]) despite some similarity in terminology and problems. This paper attempts to build a somewhat speculative bridge between two lines of network research by taking an elementary combinatorial model developed by hydrologists and geomorphologists for the analysis of stream networks (the Horton model [8]) and extending its application to include regional highway networks.

2. PATH ORDER IN THE HORTON MODEL

The sequence of regularities in the spatial structure of stream-channel networks recognized by Horton and extended by other workers (notably by Strahler and his students at Columbia University [18]) hinges on a simple combinatorial system of route ordering. The application of the system is discussed here for two types of networks: fixed-path trees (characterized by dendritic stream channels) and variable-path trees (characterized by complex highway networks).

2.1. *Combinatorial Order in Fixed-path Networks*

Melton [10] has shown that the Horton-Strahler ordering system is basically a simple mathematical concept derived from the combinatorial analysis of a finite rooted tree. Figure 1A shows such a tree in the familiar but idealized form of a channel network. The network consists of three kinds of nodes:

Set A: *Root nodes* (marked by ⊙);
Set B: *Outer nodes* (open circles) with an array of one line from each node;
Set C: *Inner nodes* (closed circles) with an array of three lines from each node.

For the network in question there is one node in Set A, eight nodes in Set B, and seven nodes in Set C. Lines joining nodes represent stream channels.

By defining any path from an outer node (Set B) as "downstream", then for any inner node (Set C) two lines are always "upstream" and only one line

* The author is Professor in the Department of Geography at the University of Bristol.